Robust full order sliding mode control for nonlinear systems under time-varying input delay

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***Abstract*—This paper proposes a systematic robust predictive control to solve the stabilization problem for a class of invertible nonlinear system with model uncertainties, exogenous disturbances and time-varying input delays. It's still an open problem to synthesize the sliding mode control (SMC) with infinite-dimensional backstepping transformation since the control input under SMC is non-differentiable. Hence, a chattering-free and Lipschitz continuous full order terminal sliding mode control is presented to enhance robustness and achieve finite-time stability simultaneously using the geometric homogeneous property. Furthermore, input-to-state stability for this target system is analyzed rigorously including internal dynamics. Finally, several simulations are illustrated to show the superiority of the proposed control comparing with the conventional state feedback laws.**

***Index Terms*—Time-varying input delay, full-order TSM, feedback linearization, robust control, Lyapunov stability.**

I. Introduction

A. History review of the literature

Control strategies for systems with input delay have been studied extensively in the last decades due to their ability of handling widespread engineering applications, for instance, industrial [1], networks [2] and robotic systems [3]. Unpredictable degradation in robustness performance and system stability may happen if the input delay is not compensated properly. Therefore, several milestone control frameworks have been proposed, such as the reduction method [4] for linear systems and infinite-dimensional backstepping transformation [5] for nonlinear systems. Meanwhile, vast majority of practical systems may suffer from both unknown input delay and plant parameters, which attracts more attention on how to effectively alleviate these uncertain influences recently. An adaptive output feedback control is proposed to overcome the unknown actuator delay, plant parameters and unmeasurable state for linear systems [6] while a systematic adaptive control under unknown actuator delay is investigated in [7] for nonlinear systems.

Moreover, since perturbations imposed on the input delay system cannot be compensated immediately due to the causality principle for real systems, it has been proved in [8] that there exists a fundamental limitation to the disturbance attenuation under the state feedback stabilizer, especially for the nonlinear system suffering from both model uncertainty and exogenous disturbance. This limitation is illustrated in [9] through the state feedback control which needs higher order state information. Unfortunately, for many nonlinear systems, there exists a major obstacle for stabilizer design due to the finite-time escape phenomenon which means the system tends to instability before the control reach. This phenomenon has been revealed in [5] and the arbitray large input delay can only be compensated under a forward complete assumption. However, this assumption may not hold when the unknown model structure and disturbance are imposed on the nominal dynamics which renders the system goto infinity. To cope with this intractable problem, we should consider other anti-disturbance methods to suppress the perturbation effects compared with the conventional state feedback control.

Naturally, we associate the anti-disturbance control with SMC, which is recognized as one of the most successful control methods due to its fast convergence, insensitivity to disturbances [10], and has been studied extensively for over 60 years. However, traditional SMC requires the actuator switching infinitely fast to enforce the ideal sliding motion which usually results in damage to the actuator and even excites high frequency unmodeled dynamics. In recent years, many improved methodologies such as high-order SMC [11], the terminal SMC (TSM) [12], state dependent boundary layer design [13] and the fixed-time SMC [14], have been proposed to explore the homogeneity, solve the chattering phenomenon and deal with the fixed time stability. Furthermore, in order to design the high-order SMC through a systematic way for nonlinear systems, a state-space exact linearization method [15] can be utilized due to its potential to transform the invertible complex nonlinear systems to linear cases. However, abundant systems do not satisfy this exact linearization condition but admit only partially linear forms, which means the remaining internal dynamics retains nonlinear and suffers from various perturbations directly especially under the existence of time-varying input delay.

B. Contribution and orgnization of the paper

Specifically, the main contribution of this paper can be summarized as the following aspects.

1) To attenuate the perturbation effects on nonlinear system with time-varying input delay one step further, we investigate the SMC techniques combined with the infinite-dimensional backstepping transformation compared with the state feedback laws. To the best of the author's knowledge, it's the first time in literature that the SMC based infinite-dimensional backstepping predictor is adopted to stabilize this kind of nonlinear system.
2) A continuous and nonsingular full order TSM, coined as CNTSM, is proposed to address the robust control problem with perturbations. Moreover, confronted with the false triggering problems for widespread applications under measurement noises, this Lipschitz continuous controller can drive the states into a customizable small neighborhood of the equilibrium in finite time while eliminating false swtiching phenomenon simultaneously. In summary, this approach can be regarded as a compromise between chattering suppression and strict TSM.
3) The CNTSM based infinite-dimensional backstepping predictor requires no higher order state information and an admissible control input is generated for practical systems. Besides, input-to-state stability of the target system is rigorously analyzed including internal dynamics. It's shown that under the proposed control, the ultimate bound of state is eliminated further, anti-disturbance ability is enhanced and system robustness under non forward complete condition is observably improved.

The rest of the article is organized as follows. Some preliminaries and motivation is firstly introduced in Section II. Delay-free robust

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control is investigated in Section III and then extended to time-varying input delay nonlinear system in Section IV. Simulation results are illustrated in Section V before the final conclusion in Section VI.

II. Preliminaries And Motivation

A. Preliminaries

We begin with recalling some basic mathematical definitions and useful lemmas for better comprehension of this paper. Definitions of functions $\mathcal{K}$, $\mathcal{KL}$ and $\mathcal{K}_\infty$ used in this paper can be found in [16]. Now consider the following full order nonlinear system:

$$\dot{x}(t) = f(x(t), u(t)) \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a Lipschitz continuous mapping function in domain $\mathcal{D} \subset R^n$, $f(0,0) = 0$ and $u \in \mathbb{R}$ is the control input.

Definition 2.1: (see [5]) Nonlinear system (1) is called strongly forward complete if there exist a smooth positive definite function R and class $\mathcal{K}_\infty$ functions $\mathit{\Pi}_1, \mathit{\Pi}_2$ and $\pi \in \mathbb{R}^+$ such that:

$$\mathit{\Pi}_1(|x|) \le R(x) \le \mathit{\Pi}_2(|x|), \quad \frac{\partial R(x)}{\partial x} f(x,u) \le R(x) + \pi(|u|)$$

for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}$. Denote $\xi(t, x_0)$ as the unique solution to system (1) from the initial state $x_0 \in \mathbb{R}^n$. This definition ensures system (1) doesn't not exhibit finite time escape phenomenon, i.e., for every initial condition and bounded input, $\xi(t, x_0)$ always exists.

Definition 2.2: (see [17]) The origin of (1) is called a finite-time-stable equilibrium if there exists an neighborhood $\mathcal{N} \subseteq \mathcal{D}$ containing the origin and a setting-time function $T : \mathcal{N}\backslash\{0\} \to \mathbb{R}^+$, such that:
(i) Finite-time convergence: For every $x \in \mathcal{N}\backslash\{0\}$ and $t \in [0, T(x))$, $\lim_{t\to T(x)} \xi(t,x) = 0$ and $\xi(t, \xi(h,x)) = \xi(t+h, x)$.
(ii) Lyapunov stability: For each open neighborhood $\mathcal{U}_\varepsilon$ of 0, there exists an open subset $\mathcal{U}_\delta$ of $\mathcal{N}$ containing 0 such that, for every $x \in \mathcal{U}_\delta\backslash\{0\}$ and $t \in [0, T(x))$, $\xi(t,x) \in \mathcal{U}_\varepsilon$.
According to the definition above, it is easy to obtain $T(x) = \inf\left\{t \in \mathbb{R}^+ : \xi(t,x) = 0\right\}$. Moreover, it can be further proved that for every $l_0 > 0$, there exists an open neighborhood $\mathcal{N}_{l_0} \subset \mathcal{N}$ including 0 such that, for every $x \in \mathcal{N}_{l_0}\backslash\{0\}$, $T(x) > l_0|x|$.

Lemma 2.1: (see [17]) For the perturbed nonlinear system $\dot{x} = f(x,u) + d(x,t)$, if there exists a positive definite and Lipschitz continuous function $V(x)$ such that $(D^+(V \circ x))(t) \le -c(V(x(t)))^\alpha + M\varrho$ for $c > 0$ and $\alpha \in (0,1)$. Then state x is ultimately bounded by (2) in finite time for $\forall t \ge \Gamma$, where $\mathit{\Gamma} = (2/c(1-\alpha))\sigma^{1-\alpha}$.

$$|x(t)| \le \frac{1}{l_0 c(1-\alpha)} \sigma^{(1-\alpha)}, \quad V(x(t)) \le \sigma \triangleq (2M\varrho/c)^{\frac{1}{\alpha}} \tag{2}$$

where $(D^+V)(t)$ is the upper right Dini derivative of V, $M > 0$ is the Lipschitz constant of V, $\gamma = (1-\alpha)/\alpha$ and the continuous disturbance $d \in \mathbb{R}$ is bounded by $\varrho = \sup_{\mathcal{D}\times\mathbb{R}^+} |d(x,t)| < \varrho_0$.

Definition 2.3: (see [18]) A family of dilations δ_ε^r is a mapping that assigns to every real $\varepsilon \in \mathbb{R}^+$ a diffeomorphism, $\delta_\varepsilon^r(x_1, \ldots, x_n) = (\varepsilon^{r_1}x_1, \ldots, \varepsilon^{r_n}x_n)$, $x \in \mathbb{R}^n$ and $r \in \mathbb{R}_n^+$ is the dilation coefficient.

A function $\mathcal{S}(x)$ is homogeneous of degree $\alpha > 0$ with respect to the family of dilations δ_ε^r, if $\mathcal{S}(\varepsilon^{r_1}x_1, \ldots, \varepsilon^{r_n}x_n) = \varepsilon^\alpha \mathcal{S}(x)$.

A vector field $f(x) = (f_1(x), \ldots, f_n(x))^{\mathrm{T}}$ is homogeneous of degree $h \in \mathbb{R}$ with respect to the family of dilations δ_ε^r if $f_i(\varepsilon^{r_1}x_1, \ldots, \varepsilon^{r_n}x_n) = \varepsilon^{h+r_i} f_i(x)$, $i = 1, \ldots, n$.

Lemma 2.2: (see [18]) Suppose the continuous vector field $f(x)$ in *definition* 2.3 is homogeneous of degree $h < 0$ w.r.t. the family of dilations δ_ε^n, $k_i (i = 1, \cdots, n)$ makes the polynomial function of ψ ($\psi \in \mathbb{R}$), $\psi^n + k_n\psi^{n-1} + \cdots + k_1$ Hurwitz. Then the equilibrium $x = 0$ of $\dot{x} = f(x)$ is globally finite-time stable.

Notation: For numbers $x, \mu \in \mathbb{R}$, we denote $\lceil x \rfloor^\mu = |x|^\mu \operatorname{sgn}(x)$ from now on for brevity, where $\operatorname{sgn}(x)$ is a signum function.

Given a function $\phi(t) : \mathbb{R}^+ \to \mathbb{R}$, we denote $\phi^{-1}(t)$ as its inverse function and $\phi'(t)$ as its derivative. For a vector $x \in \mathbb{R}^n$, denote $|x(t)|$ as its Euclidean norm, $|\overline{w(t)}| = \sup_{x\in[0,1]} |w(x,t)|$ as the infinity norm of a scalar function $w(\cdot, t) \in \mathcal{L}_\infty[0,1]$.

Denote $y_1 \circ y_2(x) = y_1(y_2(x))$ as the composite function.

A continuous saturation function is defined as below:

$$\operatorname{sat}(x,\varphi) = x/\varphi, \ |x| < \varphi; \quad \operatorname{sat}(x,\varphi) = \operatorname{sgn}(x), \ |x| \ge \varphi \tag{3}$$

B. Motivation

Consider a general nonlinear system described by:

$$\dot{x} = f(x,t) + g(x,t)u(t - \mathcal{D}(t)) + d(x,t), \quad y = h(x,t) \tag{4}$$

where $f, g : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$, are continuous differentiable mapping function in domain $\mathcal{D} \subset R^n$, $u \in \mathbb{R}$ is control input, $h : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$ indicates the output function and $d : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is the exogenous disturbance. Besides, $\mathcal{D}(t) > 0$ is the time-varying input delay, denote $\phi(t) = t - \mathcal{D}(t) > 0$ and make following assumptions.

Assumption 2.1: (see [19]) Function $\phi : \mathbb{R}^+ \to \mathbb{R}$ is continuous differentiable, $\phi'(t) > 0$ for all $t \ge 0$ and $\phi(t)$ is invertible ($\phi^{-1}(t)$ exists) such that the control signal is able to reach the plant without changing the direction of propagation. Following relationships are kept valid which means $\mathcal{D}(t)$ and $\mathcal{D}'(t)$ are bidirectionally bounded.

$${\pi_0}^* = \frac{1}{\sup_{\tau\ge\phi^{-1}(0)}(\tau - \phi(\tau))}, {\pi_1}^* = \frac{1}{\sup_{\tau\ge\phi^{-1}(0)}(\phi'(\tau))} \tag{5}$$

According to [8], perturbations in (4) cannot be compensated directly due to the delayed control input. And in [9], the ISS stabilisation for nonlinear system with input delay and disturbance is investigated through conventional state feedback control. Based on these investigations, we intend to figure out the limitation of disturbance attenuation under some other robust control methods, such as SMC. Nevertheless, it's well known that the robustness of SMC can be attributed to the discontinuous switching signal which may lead to severe chattering phenomenon especially under delayed control input and measurement noise. Therefore, in this paper we systematically discuss the robustness performance of both internal and external dynamics w.r.t. (4) utilizing CNTSM based infinite-dimensional backstepping predictor and not only exogenous disturbances but also time-varying model uncertainties exist.

III. Delay-Free Nonlinear Robust Control

In this section we will discuss two control techniques to systematically solve the delay-free stabilization problem of (4) which paves the way for addressing the input delayed case in Section IV.

A. Feedback linearization of SISO objects

Assumption 3.1: We make the following assumptions in this paper:
1) Either the state space exact linearization problem of (4) is solvable (see [15]) or (4) has relative degree r in domain $\mathcal{D}$.
2) All the state variables $(x_1, \cdots, x_n)$ are measureable.
3) Exogenous disturbance is globally continuously differentiable and bounded, denote $\sup_{t\ge t_0} \|d(t)\|_2 = \bar{d}$.

An input-affine nonlinear system is called to have a relative degree r in domain $\mathcal{D}$, if the Lie derivative $L_g L_f^i h(x) = 0$ is valid for all $i < r-1$ and $L_g L_f^{r-1} h(x) \ne 0$. Moreover, according to *proposition 4.1.3* described in [15], if r is strictly less than n for every $x_0 \in \mathcal{D}$, then there exists a neighborhood $\mathcal{N}$ of x_0, and it's always possible to find $n - r$ smooth functions $\phi_{r+1}(x), \ldots, \phi_n(x)$ exist such that (6) is satisfied and the map $\mathit{\Phi}(x) = \operatorname{col}(\zeta, \eta)$ is a diffeomorphism.

$$L_g\phi_i(x) = 0, \text{for } r+1 \le i \le n, \ \forall x_0 \in \mathcal{N} \tag{6}$$

$$\operatorname{col}(\zeta, \eta) = \operatorname{col}(h(x), \ldots, L_f^{r-1}h(x), \phi_{r+1}(x), \ldots, \phi_n(x)) \tag{7}$$

Next, define $z = \varPhi(x) = \mathrm{col}\,(\zeta, \eta)$ as the new coordinate transformation, where ζ and η are external and internal dynamics respectively. Furthermore, $\dot{\zeta}_r = L_f^r h \circ \varPhi^{-1}(\zeta, \eta) + L_g L_f^{r-1} h \circ \varPhi^{-1}(\zeta, \eta) u$ is satisfied so that the request for relative degree can be fullfilled.

Finally, the nonlinear system under model uncertainties and exogenous disturbances in (4) can be transformed into a new coordinate

$$\begin{aligned} &\dot{\zeta}_{i-1} = \zeta_i,\ i = 2, \cdots, r \qquad (8)\\ &\dot{\zeta}_r = \alpha_0(\zeta, \eta) + \beta_0(\zeta, \eta) u + \tilde{\alpha}(\zeta, \eta) + \tilde{\beta}(\zeta, \eta) u + \varrho_r(z)\\ &\dot{\eta} = \partial_x \varPhi_{[r+1,n]}(f(x,t) + d(x,t))|_{x=\varPhi^{-1}(z)} = \varpi(\zeta, \eta, d) \end{aligned}$$

where $\alpha(\zeta, \eta) = L_f^r h(x)$ and $\beta(\zeta, \eta) = L_g L_f^{r-1} h(x)$ are actual nonlinear terms due to the model uncertainties while $\alpha_0(\zeta, \eta)$, $\beta_0(\zeta, \eta)$ are the nominal ones. From now on, we combine the residual terms $\tilde{\alpha}(\zeta, \eta)$, $\tilde{\beta}(\zeta, \eta) u$ as $\delta_m(z, u)$ for brevity. Moreover, term $\varrho_r(z)$ is induced by additive disturbance $d(x, t)$.
Reformulate dynamics (8) in a neat brunovsky canonical form as:

$$\begin{aligned} &\dot{\eta} = \varpi(\zeta, \eta, d), \quad y = C_c \zeta \qquad (9)\\ &\dot{\zeta} = A_c \zeta + B_c(\beta_0(\zeta, \eta) u + \alpha_0(\zeta, \eta)) + \bar{\varrho}(z, u)\\ &u = (-L_f^r h(x) + \sum\nolimits_{i=1}^{r} k_{i-1} L_f^{(i-1)} h(x))/(L_g L_f^{r-1} h(x)) \quad (10) \end{aligned}$$

where $\zeta \in R^r, \eta \in R^{n-r}$, $\bar{\varrho}(z, u) = \delta_m(z, u) + \varrho_r(z)$ is the lumped perturbation and (A_c, B_c, C_c) is the canonical representation about a chain of r integrators. The control input u makes the polynomial function $\zeta^{r-1} + k_{r-1}\zeta^{r-2} + \cdots + k_1 \zeta + k_0$ Hurwitz.

Assumption 3.2: $\dot{\eta} = \varpi(\zeta, \eta, d)$ is continuously differentiable with (ζ, η, d) and $\dot{\eta} = \varpi(0, \eta, 0)$ is exponentially stable in a neighborhood of $\eta = 0$ denoted as $\mathcal{D}_{\eta 1} = \{\eta : \ 0 \le \|\eta\|_2 \le \eta_M\}$. Moreover, $\beta_0(x)$ is Lipschitz continuous for $\forall x \in \mathcal{D}$ with a corresponding constant L_β.

Proposition 1: If $\|\delta_m(z, u)\|_2 \le k_\vartheta \|z\|_2 + \delta_\vartheta$ and $\|\varrho_r(z)\|_2 \le \bar{\varrho}_\vartheta$ are satisfied for some constant $k_\vartheta \in \mathbb{R}^+$ and $\forall z \in \mathbb{R}^n$, $\dot{\eta} = \varpi(\zeta, \eta, d)$ satisfies the properties in *Assumption* 3.2, then under feedback control u, the state $z = \mathrm{col}(\zeta, \eta)$ is ultimately bounded by a class $\mathcal{K}$ function of δ_ϑ, $\bar{\varrho}_\vartheta$ only when $\eta(t_0) \in \mathcal{D}_{\eta 2}$ together with a norm-limited perturbation, where $\mathcal{D}_{\eta 2} \subset \mathcal{D}_{\eta 1}$.

Remark 3.1: The complete proof of *Proposition 1* is given in Appendix I and its robustness performance will be compared with the sliding mode control in the next subsection.

B. Continuous nonsingular terminal sliding mode control

The mission of the CNTSM is to force the system slide along a prescribed sliding manifold and then compel all the state variables converge into a customizable vicinity of equilibrium in finite time through a Lipschitz continuous manner. It's worth noting that this CNTSM method is inspired by [12] to obtain insensitive performance under measurement noises, which will be illustrated in Section V.

Consider the full order nonlinear system described in (9), to accomplish finite time convergence of the dynamics, the sliding manifold is designed as:

$$s = \ \dot{\zeta}_r + k_r \lceil \zeta_r \rfloor^{\mu_r} + \cdots + k_i \lceil \zeta_i \rfloor^{\mu_i} + \cdots + k_1 \lceil \zeta_1 \rfloor^{\mu_1} = 0 \quad (11)$$

where $k_i, i = 1, \cdots, r-1$ makes the polynomial function of ψ ($\psi \in \mathbb{R}$), $\psi^r + k_r \psi^{r-1} + \cdots + k_2 \psi + k_1$ Hurwitz. Besides, when the ideal sliding manifold $s = 0$ is reached, μ_i can be defined satisfying the following conditions:

$$\mu_{i-1} = \mu_i \mu_{i+1}/(2\mu_{i+1} - \mu_i),\ i = 2, \ldots, r,\ \forall r \ge 2 \qquad (12)$$

where $\mu_{r+1} = 1,\ \mu_r = \mu,\ \mu \in (0, 1)$ and $\mu_1 = \mu$ when $r = 1$.

Remark 3.2: We can prove that $0 < \mu_i < 1,\ i, j = 1, \ldots, r$.

step 1: When $i = r$, we can obtain $\mu_r = \mu < \mu_{r+1} = 1$.

step 2: When $i = r - j$, suppose $\mu_{r-j} < \mu_{r-j+1}$ is valid.

step 3: When $i = r - j - 1$, using (12) yields

$$\mu_{r-j-1} = \frac{\mu_{r-j}\mu_{r-j+1}}{2\mu_{r-j+1} - \mu_{r-j}} = \frac{\mu_{r-j}\mu_{r-j+1}}{\mu_{r-j+1} + (\mu_{r-j+1} - \mu_{r-j})}$$

We can further obtain $\mu_{r-j-1} < (\mu_{r-j}\mu_{r-j+1})/(\mu_{r-j+1}) = \mu_{r-j}$ and using the induction method, it can be concluded that $0 < \mu_1 < \cdots < \mu_r < 1$ is valid. Now we begin to prove the stability and robustness of system (9) under the proposed CNTSM.

Theorem 1: The sliding manifold of nonlinear system (9) will reach into a customizable small neighborhood of $s = 0$ in finite time and then both internal and external states $z = \mathrm{col}(\zeta, \eta)$ converge into the vicinity of equilibrium along the manifold within finite-time under *Assumption* 3.2, if the sliding manifold s is selected as (11) and the Lipschitz continuous control is designed as follows:

$$\begin{aligned} &u = \beta_0^{-1}(\zeta, t)\,(u_{eq} + u_{sw}) \qquad (13)\\ &u_{eq} = -\alpha_0(\zeta, \eta) - \sum\nolimits_{i=1}^{r} k_i\, \mathrm{sat}(\zeta_i, \varphi_i) |\zeta_i|^{\mu_i} \qquad (14)\\ &\mathcal{T} \dot{u}_{sw} + u_{sw} = -\eta^\star \,\mathrm{sgn}(s) = -(k_p + k_u + \lambda)\,\mathrm{sgn}(s) \qquad (15) \end{aligned}$$

function $\mathrm{sat}(\zeta, \varphi)$ is defined in (3), $u_{sw}(t_0) = 0; k_i$ and $\mu_i (i = 1, \ldots, r)$ are constants that defined in (11) and (12); constants $\mathcal{T}, \lambda > 0$ and k_p, k_u satisfying $k_p \ge |\dot{\bar{\varrho}}(z, u)|_{\max}, k_u \ge |u_{sw}(t)|_{\max}$.

Proof: Substituting the control (13) in (11) yields:

$$\begin{aligned} &s(t) = u_{sw} + \bar{\varrho}(z, u) + \varrho_0(t) = s_1(t) + \varrho_0(t) \qquad (16)\\ &\varrho_0(t) = \sum\nolimits_{i=1}^{r} k_i\,(\mathrm{sgn}(\zeta_i) - \mathrm{sat}(\zeta_i, \varphi_i)) |\zeta_i|^{\mu_i} \qquad (17) \end{aligned}$$

and $\bar{\varrho}_0 = \sup_{\bigcap |\zeta_i| < \varphi_i} \varrho_0(t)$. It should be noticed that the control law defined in (14) is Lipschitz continuous since the i-th control derivative w.r.t. ζ equals: $\dot{u}_{eq-i} = k_i(\mu_i + 1)\lceil \zeta_i \rfloor^{\mu_i}/\varphi_i,\ |\zeta_i| \le \varphi_i$. As a comparision, u_{eq} defined in [12] only satisfies Hölder continuity when $|\zeta_i| \to 0$ and will behave singularly especially under the existence of state measurement noises. The solution of $\dot{u}_{sw}$ is calculated as (18), thus $u_{sw}(t)$ is also continuously differentiable, both $|u_{sw}|$ and $|\dot{u}_{sw}|$ are bounded (refer to (15)) and $|\dot{u}_{sw}|$ will decrease as the increasement of parameter $\mathcal{T}$.

$$\dot{u}_{sw}(t) = -(u_{sw}(t_0) - \eta^\star \,\mathrm{sgn}(s)) e^{-\frac{t-t_0}{\mathcal{T}}}/\mathcal{T} \qquad (18)$$

Now choose a Lyapunov candidate as $V_{s1} = {s_1}^2/2$ (see (16)), then its derivative w.r.t. time t can be derived as

$$\begin{aligned} \dot{V}_{s1} &= s_1 \dot{s_1} = (\dot{\bar{\varrho}}(z, u)s - \frac{k_p}{\mathcal{T}}|s|) - (\frac{u_{sw}}{\mathcal{T}} s + \frac{k_u}{\mathcal{T}}|s|) - \frac{\lambda}{\mathcal{T}}|s|\\ &- (\dot{\bar{\varrho}}(z, u) - \frac{\eta^\star}{\mathcal{T}}\,\mathrm{sgn}(s) - \frac{u_{sw}}{\mathcal{T}})\varrho_0(t)\\ &\le -\frac{\lambda}{\mathcal{T}} V_{s1}^{1/2} + \bar{\mathcal{S}}_0, \quad \bar{\mathcal{S}}_0 = (k_p + \frac{k_u + \eta^\star}{\mathcal{T}})\bar{\varrho}_0 \qquad (19) \end{aligned}$$

In the first stage, according to *Lemma* 2.1 we can conclude that the sliding manifold $s(t)$ defined in (16) can only converge into a neighborhood of $s_1 = 0$ within finite time and its ultimate bound is

$$\begin{aligned} &|s_1(t)| \le 4\mathcal{T}^2 \bar{\mathcal{S}}_0/(\lambda^2 l_0) = \bar{\varsigma}_0 \bar{\varrho}_0,\ t \ge \varGamma \qquad (20)\\ &\bar{\varsigma}_0 = 4\mathcal{T}^2(k_p + \frac{k_u + \eta^\star}{\mathcal{T}})/(\lambda^2 l_0),\ \varGamma = 8\mathcal{T}^2 \bar{\mathcal{S}}_0/\lambda^2 \qquad (21) \end{aligned}$$

where $\bar{\varrho}_0$ (obtained when every ζ_i arrive into their layers) is defined in (17) and l_0 is a constant w.r.t. the initial value of $\zeta(t_0)$ in (2). Finally the sliding manifold $s(t)$ defined in (11) is ultimately bounded by

$$s(t) = (\varsigma_0(t) + 1)\varrho_0(t) \le (\bar{\varsigma}_0 + 1)\bar{\varrho}_0,\ t \ge \varGamma \qquad (22)$$

In the second stage, external dynamics in system (9) is homogeneous of degree $\mu - 1 < 0$ (see *Lemma* 3.1 later). According to *Lemma* 2.2, since gain k is chosen to make the polynomial function of $s = 0$ in (11) Hurwitz, state ζ will converge into a small vicinity of equilibrium along $s(t)$ in finite time Γ_1 under (17) and (22), since the bounded value $(\bar{\varsigma}_0+1)\bar{\varrho}_0$ can be regarded as an inhomogeneous term of (11). Consequently, there exists a class $\mathcal{KL}$ function β_{s1} and $\Gamma \in \mathbb{R}^+$ equals (21) such that ζ is ultimately bounded by β_{s1} and a $\mathcal{K}$ function $\mathcal{E}_{s1}$ within finite time through the same proof as in [20].

$$\|\zeta(t)\| \leq \beta_{s1}(\|\zeta(t_0)\|, t-t_0),\ t_0 \leq \forall t \leq t_0+\Gamma+\Gamma_1 \tag{23}$$

$$\|\zeta(t)\| \leq \mathcal{E}_{s1}((\bar{\varsigma}_0+1)\bar{\varrho}_0) \triangleq \bar{\gamma}_{sat1},\ \forall t > t_0+\Gamma+\Gamma_1 \tag{24}$$

Analogously, the total dynamics is ultimately bounded by (25) with $\bar{\chi}_{s1} \triangleq p_4 L(\bar{\gamma}_{sat1}+\bar{d})/(p_3(1-\iota_2))$ (details see *Proposition* 1).

$$\|z(t)\|_2 \leq \bar{\gamma}_{sat1} + \iota_3\bar{\chi}_{s1} + \sigma_1^{-1}(\sigma_2(\bar{\chi}_{s1})) \tag{25}$$

The ultimate bound of z under CNTSM is obviously smaller compared with (96) since the customizable small term $(\bar{\varsigma}_0+1)\bar{\varrho}_0$ is independent of the perturbation $\bar{\varrho}(z,u)$ in (9) such that $\bar{\Upsilon}_{pert}$ (see (89)) in both internal and external dynamics is overcomed. Thus, the proof of *theorem* 1 is completed and robust analysis above can be extended to a global concept if $\mathcal{D}_{\eta 1}$ in *Assumption* 3.2 equals $\mathbb{R}^n$.

Moreover, according to *Assumption* 3.2, the converse Lyapunov theorem (see [16] and (7)), we can deduce that there always exists a Lyapunov function V_{t1} to describe the dynamics (9) under CNTSM.

$$V_{t1}(x) = V_{s1}(\zeta) + V_{\eta 2}(\eta),\ \alpha_3(|x|) \leq V_{t1}(x) \leq \alpha_4(|x|) \tag{26}$$

$$\dot{V}_{t1}(x(t)) \leq -\rho_1(|x(t)|) + \rho_2(\bar{\chi}_{s1}) \tag{27}$$

where $V_{\eta 2}$ is defined in (90) and $\rho_1, \rho_2, \alpha_3, \alpha_4 \in$ class $\mathcal{K}_\infty$.

Remark 3.3: There are two facts worth noting with respect to the proposed CNTSM law.

1) The proof above implies the arrival sequence of ζ_i into their corresponding layers is inconsequential under the designed sliding manifold (11) and control policy (13).
2) Sliding manifold defined in (11) is unavailable because of the higher order term $\dot{\zeta}_r$. Nevertheless, this won't effect the final results since only sgn(s) is required as in (15).

Denote $g(t) = \zeta_r(t) - \zeta_r(0) + \int_0^t (\sum_{i=1}^r k_i \lceil \zeta_i \rfloor^{\mu_i})$ and ts as the system sampling time, then sgn(s) $=$ sgn$(g(t) - g(t-ts))$.

Lemma 3.1: Consider one full order sliding manifold as defined in (11) with order n and $\mu_i (i=1,\cdots,n)$ in (12), then the vector field $\mathscr{E}_\nu$ under control (13) is homogeneous of degree $\mu-1$ w.r.t. the family of dilations δ_ε^r, if r_i is selected as $r_i = (1-\mu)(n-i)+1$.

proof: The closed loop dynamics under the feedback control (13) is regulated as a chain of n integrators (8) with the respective vector $\mathscr{F}(\zeta) = (\mathscr{F}_1(\zeta), \ldots, \mathscr{F}_n(\zeta))^{\mathrm{T}}$. Therefore, denote the Euler vector field as $\mathscr{E}_\nu$ which satisfies $\mathscr{E}_\nu = \sum_{i=1}^n \mathscr{F}_i(\zeta_1,\cdots,\zeta_n)\frac{\partial}{\partial \zeta_i}$. Then it's easy to obtain following relationships of the vector $\mathscr{F}$.

$$\begin{aligned}&\mathscr{F}_i(\varepsilon^{(1-\mu)(n-1)+1}\zeta_1,\cdots,\varepsilon^{(1-\mu)(n-i)+1}\zeta_i,\cdots,\varepsilon\zeta_n) \\ &= \varepsilon^{(1-\mu)(n-i-1)+1}\zeta_{i+1} = \varepsilon^{(1-\mu)(n-i-1)+1}\mathscr{F}_i(\zeta),\ i \leq n-1\end{aligned} \tag{28}$$

$$\begin{aligned}&\mathscr{F}_n(\varepsilon^{(1-\mu)(n-1)+1}\zeta_1,\cdots,\varepsilon^{(1-\mu)(n-i)+1}\zeta_i,\cdots,\varepsilon\zeta_n) \\ &= (\varepsilon k_n\lceil\zeta_n\rfloor)^{\mu_n} + \cdots + (\varepsilon^{(1-\mu)(n-i)+1}k_i\lceil\zeta_i\rfloor)^{\mu_i} + \cdots \\ &+ (\varepsilon^{(1-\mu)(n-1)+1}k_1\lceil\zeta_1\rfloor)^{\mu_1} = \varepsilon^{\mu}\sum_{i=1}^n k_i\lceil\zeta_i\rfloor^{\mu_i} = \varepsilon^{\mu}\mathscr{F}_n(\zeta)\end{aligned} \tag{29}$$

Hence, according to *Definition* 2.3 that $\mathscr{F}_i(\varepsilon^{r_1}\zeta_1,\ldots,\varepsilon^{r_n}\zeta_n) = \varepsilon^{h+r_i}\mathscr{F}_i(\zeta)$, we can conclude the Euler vector field $\mathscr{E}_\nu$ is homogeneous of negative degree $\mu-1$ w.r.t. the dilation $\delta_\varepsilon^r(\zeta_1,\ldots,\zeta_n)$.

IV. SMC Based Predictor For Time-Varying Input Delay Nonlinear System

Reformulate the nonlinear dynamics (4) (replace x with X to avoid ambiguity in later proof) in a concise form as (30), function $\mathcal{F}(X(t),U(t),0)$ indicates the ideal nominal model and $\mathcal{F}:\mathbb{R}^n \times 1 \times \mathbb{R}^m \to \mathbb{R}^n$ is continuously differentiable to all the arguments with $\mathcal{F}(0,0,0)=0$. It's worth noting that $\Upsilon(t)$ is the lumped perturbation including model uncertainties, exogenous disturbances and their effects on internal dynamics (see *Appendix* I).

$$\dot{X}(t) = \mathcal{F}(X(t), U(t-\mathcal{D}(t)), \Upsilon(t)) \tag{30}$$

$$\Upsilon(t) = \Upsilon(\bar{\varrho}(t), d(t)) \leq \bar{\Upsilon}(\bar{\varrho}, \bar{d}) \tag{31}$$

We utilize the Lyapunov methods introduced in [21] in this section. First, transform the original nonlinear system (30) into a transport PDE and nonlinear ODE cascade system through a rescaled unity interval notation and denote $\mathcal{G}(t) = \phi^{-1}(t) - t$ from now on.

$$\dot{X}(t) = \mathcal{F}(X(t), u(0,t), \Upsilon(t)),\quad u(1,t) = U(t) \tag{32}$$

$$\partial_t u(x,t) = \pi(x,t)\partial_x u(x,t),\ x \in [0,1] \tag{33}$$

where $u(x,t) = U(\phi(\mathcal{G}(t)x+t))$ and $\pi(x,t) = (\mathcal{G}'(t)x+1)/\mathcal{G}(t)$.

Lemma 4.1: The ODE-PDE cascade system (32)-(33) can be transformed into a target system below through the infinite-dimensional backstepping transformation defined in (38).

$$\dot{X}(t) = \mathcal{F}(X(t), \psi(X(t)) + w(0,t), \Upsilon(t)) \tag{34}$$

$$\partial_t w(x,t) = \pi(x,t)\partial_x w(x,t) - \frac{\partial\psi(p(x,t))}{\partial p(x,t)}\mathcal{Q}(x,\Upsilon(t),t) \tag{35}$$

$$w(1,t) = U(t) - \psi(P(t)) = 0 \tag{36}$$

$$\mathcal{Q}(x,\Upsilon(t),t) = \partial_t p(x,t) - \pi(x,t)\partial_x p(x,t) \tag{37}$$

proof: The invertible backstepping-forwarding transformation is defined as below and function (34) can be obtained using (38).

$$w(x,t) = u(x,t) - \psi(p(x,t)),\ \forall x \in [0,1] \tag{38}$$

$$p(x,t) = P(\phi(\mathcal{G}(t)x+t)),\ \forall x \in [0,1] \tag{39}$$

function ψ is the continuous controller and $P(t)$ is the predicted state that $\mathcal{G}(t)$ time ahead of $X(t)$ based on the nominal model, thus

$$\begin{aligned}P(\tau) &= \mathcal{G}(t)\int_0^{\frac{\phi^{-1}(\tau)-t}{\mathcal{G}(t)}} \mathcal{F}(P(\phi(\mathcal{G}(t)\tau+t)), \\ &U(\phi(\mathcal{G}(t)\tau+t)),0)d\tau + X(t),\ \phi(t) \leq \tau \leq t\end{aligned} \tag{40}$$

$$p(x,t) = \mathcal{G}(t)\int_0^x \mathcal{F}(p(\tau,t),u(\tau,t),0)\mathrm{d}\tau + X(t) \tag{41}$$

Notice that $p(0,t) = X(t)$ when $x=0$ in (41). Next, differentiate (41) with respect to x and t, we can examine that

$$\begin{aligned}\partial_t p(x,t) &= \mathcal{G}(t)\int_0^x \frac{\partial\mathcal{F}(p(\tau,t),u(\tau,t),0)}{\partial p}\partial_t p(\tau,t) + \mathcal{G}(t) \\ &\times \int_0^x \frac{\partial\mathcal{F}(p(\tau,t),u(\tau,t),0)}{\partial u}\pi(\tau,t)\partial_\tau u(\tau,t)d\tau + \mathcal{G}'(t) \\ &\times \int_0^x \mathcal{F}(p(\tau,t),u(\tau,t),0)d\tau + \mathcal{F}(p(0,t),u(0,t),\Upsilon(t))\end{aligned} \tag{42}$$

$$\begin{aligned}\pi(x,t)\partial_x p(x,t) &= \mathcal{G}(t)\pi(x,t)\mathcal{F}(p(x,t),u(x,t),0) \\ &= \mathcal{G}(t)\int_0^x d(\pi(\tau,t)\mathcal{F}(p(\tau,t),u(\tau,t),0)) \\ &+ \mathcal{G}(t)\times\pi(0,t)\mathcal{F}(p(0,t),u(0,t),0)\end{aligned} \tag{43}$$

Utilizing the relationships above, the solution of $\mathcal{Q}(x,\Upsilon(t),t)$ to the homogeneous differential equation (44) with (45) is given by (46), where exp represents the function with natural logarithm.

$$\partial_x\mathcal{Q}(x,\Upsilon(t),t) = \mathcal{G}(t)\frac{\partial\mathcal{F}(p(x,t),u(x,t),0)}{\partial p(x,t)}\mathcal{Q}(x,\Upsilon(t),t) \tag{44}$$

$$\mathcal{Q}(0,\Upsilon(t),t) = \mathcal{F}(X(t),u(0,t),\Upsilon(t)) - \mathcal{F}(X(t),u(0,t),0) \tag{45}$$

$$\frac{\mathcal{Q}(x,\Upsilon(t),t)}{\mathcal{Q}(0,\Upsilon(t),t)} = \exp(\,\mathcal{G}(t)\int_0^x \frac{\partial\mathcal{F}(p(\tau,t),u(\tau,t),0)}{\partial p(\tau,t)}d\tau\,) \tag{46}$$

Take partial derivative of $w(x,t)$ in (38) with t and using (33) yields

$$\begin{aligned}\partial_t w(x,t) &= \partial_t u(x,t) - \partial_t \psi(p(x,t)) \qquad (47)\\ &= \pi(x,t)(\partial_x w(x,t) + \partial_x \psi(p(x,t))) - \partial_t \psi(p(x,t))\\ &= \pi(x,t)\partial_x w(x,t) - \frac{\partial \psi(p(x,t))}{\partial p(x,t)} \mathcal{Q}(x, \varUpsilon(t), t)\end{aligned}$$

Thus, *Lemma* 4.1 is proved.
Meanwhile, the inverse forwarding transformation is defined as:

$$\begin{aligned}u(x,t) &= \psi(\tilde{p}(x,t)) + w(x,t),\ \forall x \in [0,1] \qquad (48)\\ \tilde{p}(x,t) &= \mathcal{G}(t)\int_0^x \mathcal{F}(\tilde{p}(\tau,t), \psi(\tilde{p}(\tau,t)) + w(\tau,t), 0)d\tau + X(t)\end{aligned}$$

Under the forwarding transformation (48), the target system (34)-(36) is transformed back to system (32)-(33).

Next, we introduce several inequalities and the infinity norm described in Section II is used. According to (13), the CNTSM controller we designed is Lipschitz continuous. Hence, the following inequalities are valid where $\kappa_0, \kappa_1 \in$ class $\mathcal{K}_\infty$.

$$|\psi(p(t))| \leq \kappa_0(|\overline{p(t)}|), \quad |\frac{\partial \psi(p(t))}{\partial p(t)}|^2 \leq \kappa_1(|\overline{p(t)}|) \qquad (49)$$

$$\varXi_1(t) = |X(t)| + |\overline{u(t)}|, \quad \varXi_2(t) = |X(t)| + |\overline{w(t)}| \qquad (50)$$

Under the conditions in (49), we can further obtain that

$$|\overline{p(t)}| \leq \varOmega_1(|\varXi_1(t)|), \quad |\overline{w(t)}| \leq |\overline{u(t)}| + \varOmega_3(|\varXi_1(t)|) \qquad (51)$$

$$|\overline{\tilde{p}(t)}| \leq \varOmega_2(|\varXi_2(t)|), \quad |\overline{u(t)}| \leq |\overline{w(t)}| + \varOmega_4(|\varXi_2(t)|) \qquad (52)$$

where $\varOmega_i (i = 1, \cdots, 4) \in$ class $\mathcal{K}_\infty$ and (51)-(52) have been proved in [21]. Moreover, according to *Assumption* 3.2, function $\mathcal{F}$ is continuously differentiable w.r.t. all the arguments. Thus we can obtain the following relationships using (31) and (45).

$$|\frac{\partial(\mathcal{F}(p(x,t), u(x,t), 0))}{\partial p(x,t)}| \leq \kappa_2(|\overline{p(t)}|) + \kappa_3(|\overline{u(t)}|) \qquad (53)$$

$$\mathcal{Q}(0, \varUpsilon(t), t)^2 = (\,\varUpsilon(t) \times \int_0^1 (\frac{\partial \mathcal{F}(X(t), u(0,t), \tau\varUpsilon(t))}{\partial(\tau\varUpsilon(t))})d\tau\,)^2$$

$$\mathcal{Q}(0, \varUpsilon(t), t)^2 \leq \bar{\varUpsilon}^2(\,\varTheta_1(|X(t)|) + \varTheta_2(|\overline{u(t)}|) + \varTheta_3(\bar{\varUpsilon})\,) \qquad (54)$$

where $\kappa_2, \kappa_3 \in$ class $\mathcal{K}_\infty$ and $\varTheta_i (i = 1, \cdots, 3) \in$ class $\mathcal{K}$.

Theorem 2: The ISS property of original system (32)-(33) and target system (34)-(36) can be achieved under the CNTSM control defined in (13) and *Assumption* 2.1, 3.1, 3.2. There exists $\beta_{smc1}, \beta_{smc2} \in$ class $\mathcal{KL}$ and $\mathcal{E}_{smc1}, \mathcal{E}_{smc2} \in$ class $\mathcal{K}$ such that the following relationships hold for $\forall t \geq t_0$.

$$\varXi_1(t) \leq \beta_{smc1}(\varXi_1(0), t) + \mathcal{E}_{smc1}(\bar{\varUpsilon}, \bar{\varrho}_0) \qquad (55)$$

$$\varXi_2(t) \leq \beta_{smc2}(\varXi_2(0), t) + \mathcal{E}_{smc2}(\bar{\varUpsilon}, \bar{\varrho}_0) \qquad (56)$$

Proof: Consider the sliding manifold s defined in (11) and substitute the backstepping controller $\psi(X(t)) + w(0,t)$ into the original nonlinear system (9), where controller ψ is defined in (13). Then $s = u_{sw} + \bar{\varrho}(z,u) + \varrho_0(t) + \beta_0(X)w(0,t)$, denote $V_{s2} = {s_2}^2/2$ as the ISS Lyapunov function for (34)-(36) and $s_2 = u_{sw} + \bar{\varrho}(z,u)$.

$$\begin{aligned}\dot{V}_{s2} &= \left(\dot{\bar{\varrho}}(z,u)s - \frac{k_p}{\mathcal{T}}|s|\right) - \left(\frac{u_{sw}}{\mathcal{T}}s + \frac{k_u}{\mathcal{T}}|s|\right) - \frac{\lambda}{\mathcal{T}}|s| \qquad (57)\\ &- (\dot{\bar{\varrho}}(z,u) - \frac{\eta^\star}{\mathcal{T}}\operatorname{sgn}(s) - \frac{u_{sw}}{\mathcal{T}})(\varrho_0(t) + \beta_0(X)w(0,t))\\ &\leq -\frac{\lambda}{\mathcal{T}}V_{s2}^{1/2} + \bar{\mathcal{S}}_d,\ \bar{\mathcal{S}}_d = (k_p + \frac{k_u + \eta^\star}{\mathcal{T}})(\bar{\varUpsilon}_w + \bar{\varrho}_0)\end{aligned}$$

Then the ultimate bound of $s(t)$ can be calculated the same way as *Theorem* 1 and denote $\bar{\varUpsilon}_w = \sup_{\phi(t)\leq\tau\leq t} |\beta_0(X(\tau))w(0,\tau)|$.

$$|s(t)| \leq (\bar{\varsigma}_d + 1)(\bar{\varUpsilon}_w + \bar{\varrho}_0),\ \bar{\varsigma}_d = \frac{4\mathcal{T}^2}{\lambda^2 l_0}(k_p + \frac{k_u + \eta^\star}{\mathcal{T}}) \qquad (58)$$

$$|s_2(t)| \leq 4\mathcal{T}^2\bar{\mathcal{S}}_d/(\lambda^2 l_0),\ t \geq \varGamma_d \triangleq 8\mathcal{T}^2\bar{\mathcal{S}}_d/\lambda^2 \qquad (59)$$

Consequently, there exists a class $\mathcal{K}_\infty$ function $\mathcal{E}_{s2}$ w.r.t. $\bar{\varUpsilon}_w$ such that external state ζ is ultimately bounded within finite time (an essential remark will be given in *Remark* 4.1 after the proof of *Theorem* 2).

$$\|\zeta(t)\| \leq \mathcal{E}_{s2}((\bar{\varsigma}_d + 1)(\bar{\varUpsilon}_w + \bar{\varrho}_0)) \triangleq \bar{\varUpsilon}_{sat2},\ \forall t > t_0 + \varGamma_d + \varGamma_{d1}$$

Analogously, the total dynamics is ultimately bounded by (60) with $\bar{\chi}_{s2} \triangleq p_4 L(\bar{\varUpsilon}_{sat2} + \bar{d})/(p_3(1 - \iota_2))$(details see *Proposition* 1).

$$\|z(t)\|_2 \leq = \bar{\varUpsilon}_{sat2} + \iota_3\bar{\chi}_{s2} + \sigma_1^{-1}(\sigma_2(\bar{\chi}_{s2})) \qquad (60)$$

In summary, a Lyapunov function $V_{t2}(X)$ can be selected in the same way as (26)-(27) to describe (34) and $\rho_3, \rho_4, \rho_5, \rho_6, \rho_w, \alpha_5, \alpha_6 \in$ class $\mathcal{K}_\infty$, and the Young's Inequality of $\bar{\varUpsilon}_w$ (above (58)) is used.

$$\begin{aligned}&V_{t2}(X) = V_{s2}(\zeta) + V_{\eta 2}(\eta),\ \alpha_5(|X|) \leq V_{t2}(X) \leq \alpha_6(|X|) \qquad (61)\\ &\dot{V}_{t2}(X) \leq -\rho_3(|X(t)|) + \rho_4(\bar{\varrho}_0, \bar{d}) + \rho_w(\bar{\varUpsilon}_w) \qquad (62)\\ &\leq -\rho_3(|X(t)|) + \rho_4(\bar{\varrho}_0, \bar{d}) + \rho_5(\beta_0^2(X(t))) + \rho_6(w^2(0,t))\end{aligned}$$

Next, select a Lyapunov function $L(t)$ w.r.t. $w(x,t)$ and $h \in \mathbb{R}^+$.

$$L(t) = \frac{h}{2}\int_0^1 e^{cx}w^2(x,t)dx,\ c \geq (1 - \pi_1^*)\max\{1, 1/\pi_1^*\} \qquad (63)$$

$$\begin{aligned}\dot{L}(t) &= \frac{h}{2}\int_0^1 e^{cx}\pi(x,t)dw^2(x,t) \qquad (64)\\ &- h\int_0^1 e^{cx}w(x,t)(\pi(x,t)\partial_x w(x,t) - \partial_t w(x,t))dx\\ &= -\frac{h}{2}\int_0^1 (c\pi(x,t) + \partial_x\pi(x,t))e^{cx}w^2(x,t)dx \qquad (65)\\ &- \frac{h\pi(0,t)}{2}w^2(0,t) - h\int_0^1 e^{cx}w(x,t)\frac{\partial\psi(p(x,t))}{\partial p(x,t)}\mathcal{Q}(x, \bar{\varUpsilon}, t)dx\end{aligned}$$

notice that $c\pi(x,t) + \partial_x\pi(x,t)$ in (65) is only linear with x, use the same proof as in [21], we get the lower bound of $c\pi(x,t) + \partial_x\pi(x,t)$.

$$\gamma_0 = \min\{c - 1 + \pi_1^*, (c+1)\pi_1^* - 1\} > 0 \qquad (66)$$

$$c\pi(x,t) + \partial_x\pi(x,t) \geq \pi_0^*\gamma_0 \qquad (67)$$

$$\varLambda(x, \bar{\varUpsilon}, t) = \frac{he^c}{\pi_0^*\gamma_0}\int_0^1 (\,\frac{\partial\psi \circ p(x,t)}{\partial p(x,t)}\mathcal{Q}(x, \bar{\varUpsilon}, t)\,)^2 dx \qquad (68)$$

According to (46) and (49)-(54), $\varLambda(x, \bar{\varUpsilon}, t)$ above is bounded by (69), we use $\circ$ introduced in section II for the convenience of reading.

$$\begin{aligned}\frac{\pi_0^*\gamma_0}{e^c}\varLambda(x, \bar{\varUpsilon}, t) &\leq \kappa_1(|\overline{p(t)}|)\exp(\,\frac{2}{\pi_0^*}(\kappa_2 \circ |\overline{p(t)}| + \kappa_3 \circ |\overline{u(t)}|\,)\,)\\ &\times h\bar{\varUpsilon}^2(\,\varTheta_1 \circ |X(t)| + \varTheta_2 \circ |\overline{u(t)}| + \varTheta_3 \circ \bar{\varUpsilon}\,)\\ &\leq h(\,\varPsi_1(\varXi_2(t))\varPsi_2(\varXi_2(t)) + \varPsi_1(\varXi_2(t))\varTheta_3(\bar{\varUpsilon})\,)\,\bar{\varUpsilon}^2 \qquad (69)\end{aligned}$$

$$\varPsi_1(y) = \kappa_1 \circ \varOmega_1(\varOmega_4'(y)) \times \exp(\frac{\kappa_2 \circ \varOmega_1 \circ \varOmega_4'(y) + \kappa_3 \circ \varOmega_4'(y)}{\pi_0^*/2})$$

$$\varPsi_2(y) = \varTheta_1 \circ y + \varTheta_2 \circ \varOmega_4'(y), \quad \varOmega_4'(y) \triangleq (y + \varOmega_4 \circ y) \qquad (70)$$

Utilizing the Young's Inequality and (69) above, it yields

$$\begin{aligned}\frac{2\pi_0^*\gamma_0}{e^c}\varLambda(x, \bar{\varUpsilon}, t) &\leq \varPsi_1^2(\varXi_2(t)) + h^2\varTheta_3^2(\bar{\varUpsilon})\bar{\varUpsilon}^4\\ &+ \varPsi_1^2(\varXi_2(t))\varPsi_2^2(\varXi_2(t)) + h^2\bar{\varUpsilon}^4 \qquad (71)\end{aligned}$$

Now we turn back to analyze $\dot{L}(t)$ in (65), utilizing the relationships above and the Young's Inequality again, it yields

$$\dot{L}(t) \leq -\frac{h\pi_0^*}{2}w^2(0,t) - \frac{h\pi_0^*\gamma_0}{4}\int_0^1 e^{cx}w^2(x,t)dx + \varLambda(x, \bar{\varUpsilon}, t)$$

Finally, select a total Lyapunov candidate as $V_T = V_{t2} + L$ (61,63)

$$\begin{aligned}\dot{V}_T(t) \le& -(\rho_3 \circ |X(t)| - \rho_5 \circ \beta_0^2 \circ X(t)) + \rho_4(\bar{\varrho}_0, \bar{d}) \quad (72)\\ &-(\frac{h\pi_0^*}{2}w^2(0,t) - \rho_6 \circ w^2(0,t)) + \frac{h^2 e^c}{2\pi_0^*\gamma_0}(\Theta_3^2 \circ \bar{\Upsilon} + 1)\bar{\Upsilon}^4\\ &-(\frac{h\pi_0^*\gamma_0}{4}\int_0^1 w^2(x,t)dx - \frac{e^c\Psi_1^2 \circ \Xi_2(t)}{2\pi_0^*\gamma_0}(\Psi_2^2 \circ \Xi_2(t) + 1))\end{aligned}$$

According to *Assumption* 3.2, we know that $\rho_5 \circ \beta_0^2 \circ X(t)$ is equivalent with $\rho_5'(L_\beta X)$ for some $\rho_5' \in \mathcal{K}_\infty$. Hence, the first term in (72) tends to be negative for some large λ designed in (57). Furthermore, since $w(x,t)$ and $\Xi_2(t)$ are defined on a finite interval w.r.t. π_0^*, it's always possible to choose a sufficient large h (see (63)) to make sure the third and fifth term in (72) are both negative.

$$\dot{V}_T(t) \le -\rho_1' \circ |\Xi_2(t)| + \frac{h^2 e^c}{2\pi_0^*\gamma_0}\rho_2' \circ \bar{\Upsilon} + \rho_4(\bar{\varrho}_0, \bar{d}) \quad (73)$$

Hence, (72) can be reformulated as (73) and there's no doubt that a larger ultimate bound will pay the price for a larger h. Furthermore, the ultimate bound is positively associated with $\bar{\Upsilon}$ (31) and inversely correlated with π_0^*(5), which means larger perturbations and longer input delay will lead to inevitable worse robustness performance. Finally, the ISS of target system (34)-(36) can be proved through the same way as in [20], which is illustrated in (56). Analogously, using relationships in (50)-(52), the ISS of original system (32)-(33) can be concluded as (55). Thus, *Theorem* 2 is proved completely.

Remark 4.1: It's worth noting w.r.t. *Theorem* 2 that could we select $s_2 = u_{sw} + \bar{\varrho}(z,u) + \beta_0(X)w(0,t)$ in (57)? Such that the term $\beta_0(X)w(0,t)$ can be compensated through SMC techniques. This motivation is inspired by the fact that $|\beta_0(X)w(0,t)|$ and its derivative w.r.t. time t is bounded in a finite set $\mathcal{D}$ (according to *Assumption* 3.2, (13) and (38)). However, the answer is negative. Notice $\dot{w}(0,t)$ derived by $\dot{s}_2$ and the definition of $w(0,t)$ in (38), term $\dot{u}_{sw}(t)$ is generated as in (18), one cannot guarantee $\dot{u}_{sw}(t)$ be compensated by term $\frac{\lambda}{\mathcal{T}}|s|$ in (57) for $\forall t > 0$ especially under long time delay conditions even $\eta^\star$ is sufficient large. Hence, we could only give a prudent ISS property in *Theorem* 2 and the essential reason behind is the causality of the delayed input control.

V. Numerical Examples

A. Delay-free case

Example 1. Consider a nonlinear pendulum dynamics as below

$$\begin{aligned}\dot{x}_1 &= x_2 \quad (74)\\ \dot{x}_2 &= -a\sin(x_1 + \delta) - bx_2 + cu(t - \mathcal{D}(t)) + d(t)\end{aligned}$$

where x_1, x_2 represent the angle and angular velocity respectively. The torque input is u and $a, \delta, b, c \in \mathbb{R}$ are physic parameters. Our goal is to stabilize the pendulum at $x_1 = 0$ and a conventional feedback control is given by $cu = a\sin(x_1 + \delta) - (k_1 x_1 + k_2 x_2)$ where k_1 and k_2 are chosen as in (10). However, the actual control is generated by $\hat{c}u = \hat{a}\sin(x_1+\delta) - (k_1x_1 + k_2x_2)$ due to uncertainties in a and c, where $\hat{a}$ and $\hat{c}$ are the estimated values. Thus, $\bar{\varrho}(z,u)$ in (9) equals below and $\mathcal{D}(t) = 0$ in this case.

$$\hat{c}\bar{\varrho}(z,u) = (\hat{a}c - a\hat{c})\sin(x_1 + \delta) - (c - \hat{c})(k_1x_1 + k_2x_2) \quad (75)$$

and parameters $k_\vartheta, \delta_\vartheta$ in *Assumption* 3.2 equal:

$$k_\vartheta = |\frac{\hat{a}c - a\hat{c}}{\hat{c}}| + |\frac{c - \hat{c}}{\hat{c}}|\sqrt{k_1^2 + k_2^2},\ \delta_\vartheta = |\frac{\hat{a}c - a\hat{c}}{\hat{c}}||\sin\delta| \quad (76)$$

Estimated parameters are $\hat{a} = \hat{c} = 1$ and the actual parameters varies as $a = 1.2 + 0.4\cos(10t)$, $c = 1 + 0.2\cos(10t)$; $b = 0.4, \delta = 5$. In this case, $k_\vartheta = 1.962, \delta_\vartheta = 0.384$ and the exogenous disturbance $d = 2\sin(1.5t)$ is imposed when $t \ge 6s$. Hence, we select the control parameters in (15) as: $k_p = 3, k_u = 3, \eta^\star = 7, \mathcal{T} = 0.5$; $k1 = 6$, $k2 = 5$; $\varphi_1 = 0.01, \varphi_2 = 0.02$ in (3) and $\mu_1 = 11/25, \mu_2 = 11/18$ in (12). Besides, a band-limited white noise with power $1e-6$ is added on the measurement, initial state is given by $x(t_0) = [1, -1]$.

Remark 5.1: As illustrated in Fig. 1, states x_1 is driven into a desired small vicinity of equilibrium under CNTSM even a destructive exogenous disturbance is imposed when $t \ge 6s$. As a comparison, a static offset of x_1 exists(see (89)) under the feedback control due to model uncertainties, and its robustness become even worse when disturbance is imposed. Moreover, notice the finite time convergence is simultaneously achieved(see (21)) under CNTSM and we give a comparison with the performance of Fixed Time SMC (see [14]). Meanwhile, an admissible control u that free of false switching is generated by CNTSM under the measurement noises in Fig. 2. As a comparison, u generated by Chattering-free TSM([12]) and Fixed Time SMC([14]) in general, only make sense in a theoretical concept.

Example 2. Consider a 3rd order nominal nonlinear dynamics as

$$\begin{aligned}\dot{x}_1 &= \frac{{x_3}^2 + 2}{5({x_3}^2 + 1)}u(t - \mathcal{D}(t)) - x_1,\ \ \dot{x}_2 = \sinh(4x_3) + x_3 \quad (77)\\ \dot{x}_3 &= \frac{\sin(x_2x_3)}{\cos(x_3) + 2} + u(t - \mathcal{D}(t)), \quad y = h(x) = x_2\end{aligned}$$

The relative degree of (77) equals 2, $\beta_0(x) = 4\cosh(4x_3)$ and $\Phi(x)$ defined in (7) equals $\mathrm{col}(x_2, \sinh(4x_3), \eta)$ where $\eta = -5x_1 + x_3 + \tan^{-1}(x_3), x_2 = \zeta_1, x_3 = \ln(\zeta_2 + (\zeta_2^2+1)^{1/2})/4 = f_0(\zeta_2)$ and $\dot{\eta} = L_f\Phi(x) = (-5\eta + f_0(\zeta_2) + \tan^{-1} \circ f_0(\zeta_2)) + \sin(\zeta_1 f_0(\zeta_2)) \times \cdots$. We omit some terms by $\cdots$, since $\zeta = 0$ for the zero dynamics $\dot{\eta} = \varpi(0, \eta, 0)$ in (9) and $\dot{\eta} = -5\eta$, which is asymptotically stable and *Assumption* 3.2 valids in this case. In simulation, we set the actual model dynamics as (78) during $t \in [0, 24]s$, the disturbance $d(t) = \sin(5t)\,\mathrm{col}(0.1, 0, 0.2)$ is imposed during $t \in [15, 24]s$, both model uncertainty and disturbance are revoked during $t \in [24, 30]s$.

$$\dot{x}_3 = \frac{0.35 + \sin(x_2x_3)}{\cos(x_3) + 2} + (0.2\sin(5t) + 1)u(t - \mathcal{D}(t)) \quad (78)$$

$$\dot{\zeta}_2 = (\alpha - \alpha_0) - \frac{\beta - \beta_0}{\beta_0}(\alpha_0 + \sum\nolimits_{i=1}^2 k_i\zeta_i) - \sum\nolimits_{i=1}^2 k_i\zeta_i \quad (79)$$

According to (8) and (79), parameters $k_\vartheta, \delta_\vartheta$ in *Assumption* 3.2 equal:

$$\begin{aligned}k_\vartheta &= |\frac{\beta - \beta_0}{\beta_0}|\sqrt{k_1^2 + k_2^2}, \quad \delta_\vartheta = |\tilde{\alpha}| + |\frac{\beta - \beta_0}{\beta_0}\alpha_0| \quad (80)\\ \frac{\beta - \beta_0}{\beta_0} &= \frac{\sin(5t)}{5},\ \ \tilde{\alpha} = \frac{\cosh(4x_3)}{5}(4\sin(5t) + \frac{7}{\cos(x_3) + 2})\end{aligned}$$

from which we can obtain a conservative bound of $k_\vartheta, \delta_\vartheta$ which depend only on x_3 that belonging to a finite set $\mathcal{D}$. We keep the control parameters same with the last case with a sampling time $ts = 2ms$ and initial state is set by $x(t_0) = [1, 1, 0.15]$.

Remark 5.2: As illustrated in Fig. 3, x_2 keeps robust and converges into the designed vicinity in finite time under CNTSM when model uncertainty and disturbance both exist for $t \in [0, 24]s$, while the feedback control method remains fragile as analyzed in *Remark* 5.1. Besides, $x1$ that related to the internal dynamics $\eta = -5x_1 + x_3 + \tan^{-1}(x_3)$ shows better anti-disturbance property under CNTSM. Moreover, we give a comparison with the conventional reduced order SMC(ROSMC), in which $s_{re} = \dot{\zeta}_r + \sum_{i=1}^r k_i\zeta_i = 0$ (also see (11)). It shows that the state convergence under ROSMC is apparently slower than CNTSM due to the fact $s_{re} = 0$ doesn't possesse the negative homogeneous degree as defined in *Lemma* 3.1. A 'switching' (indeed is continuous, see (18)) control manner is demonstrated in Fig. 4 clearly for CNTSM, ROSMC methods.

B. Input delayed case

Naturally, we wonder what robustness performance will be when input delay exists, especially using SMC techniques. Consider the

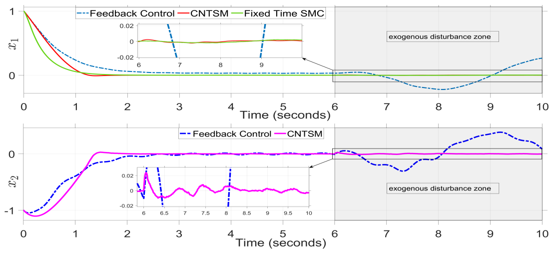


Fig. 1: State response under time-varying perturbations

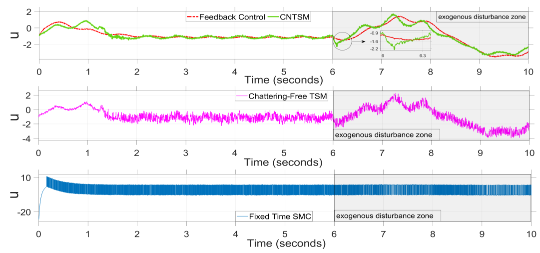


Fig. 2: Control input under time-varying perturbations

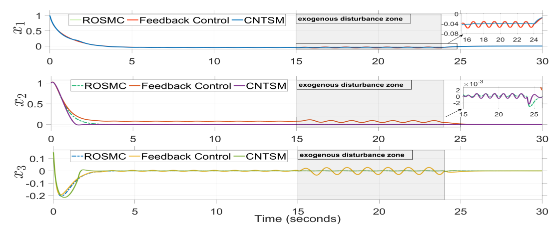


Fig. 3: State response under time-varying perturbations

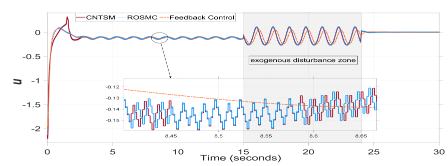


Fig. 4: Control input under time-varying perturbations

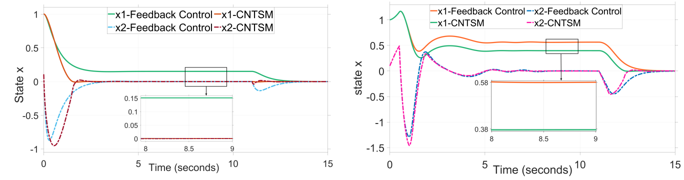
system (74) with an input delay function as (81)

$$\mathcal{D}_1(t) = \frac{t+1}{2t+1}, \quad \phi_1(t) = t - \frac{t+1}{2t+1} \tag{81}$$

$$\mathcal{G}_1(t) = \frac{t+1}{\sqrt{(t+1)^2+1}+t}, \quad \pi_0{}^* = \sqrt{2}, \quad \pi_1{}^* = 1 \tag{82}$$

we increase the model uncertainties to $a = 2+\cos(t), c = 1+0.5\cos(t)$ during $t \in [0, 11]s$ compared with the delay-free case, $x(t_0) = [1, 0.2]$ and keep the control parameters invariable.

Remark 5.3: In Fig. 5(b), there exists an ultimate bound of x_1 (see $\Xi_1(t)$ in (55)) for both two methods due to the input delay. Meanwhile, Fig. 5(b) shows that during the steady state $t \in [8, 11]s$, the difference of x_1 between CNTSM and feedback control is obviously larger than the delay-free case in Fig. 5(a), which means there exists an excess dividend for the steady state error under SMC. This fact can be explained from two aspects. First, in this case no internal dynamics exists and $\rho_4(\bar{\varrho}_0, \bar{d})$ (see (73)) that induced by model uncertainty is overcomed under CNTSM and only a customizable small term $\rho_4(\bar{\varrho}_0)$ is left. Hence, the ultimate bound of $\Xi_1(t) = |X(t)| + |\overline{u(t)}|$ under CNTSM is smaller, which is shown in Fig. 6. Secondly, $|\overline{u(t)}|$ of CNTSM is larger in steady state due to the fact that $|\zeta_i|^{\mu_i} > |\zeta_i|$ when $0 < |\zeta_i| < 1$ (see (14)) along with an extra control term u_{sw} (see (15)), this fact is also shown in Fig. 6. Finally, the excess dividend of the steady state error under SMC is explained from both



(a) state x in delay-free case (b) state x in input delayed case

Fig. 5: State response under model uncertainties

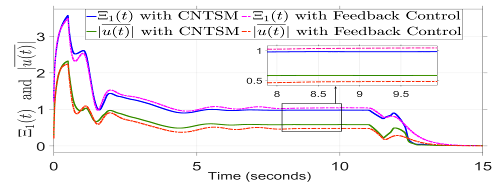
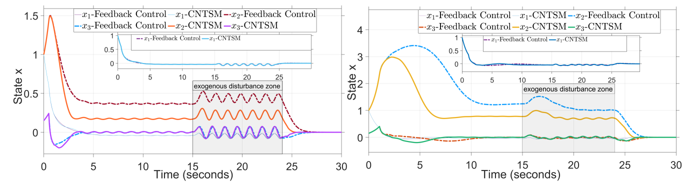


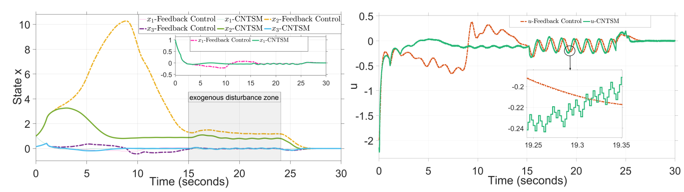
Fig. 6: $\Xi_1(t)$ and $|\overline{u(t)}|$ response under input delay case

theoretical and experimental perspectives.



(a) state x response with $\mathcal{D}_1(t)$ (b) state x response with $\mathcal{D}_2(t)$

Fig. 7: State response under model uncertainties



(a) state x response with $\mathcal{D}_2(t)$ (b) control input with $\mathcal{D}_2(t)$

Fig. 8: State response under non forward complete condition

Next, consider system (77) with another input delay function as

$$\mathcal{D}_2(t) = \frac{t+3}{t+4}, \quad \phi_2(t) = t - \frac{t+3}{t+4}, \quad \pi_0{}^* = 1 \tag{83}$$

$$\mathcal{G}_2(t) = -\frac{t+3+\sqrt{(t+3)(t+7)}}{2}, \quad \pi_1{}^* = 0.956 \tag{84}$$

and keep all the settings same with the delay-free case in (77).

Remark 5.4: According to (73) and (82,83), the ultimate bound of $x(t)$ is inversely correlated with π_0^* which has been illustrated Fig. 7. Meanwhile, the steady state error and convergence rate under CNTSM are both superior to the feedback control with an admissible control. Besides, perturbations are revoked during $t \in [24, 30]s$ and relate it with the disturbance zone in Fig. 7, one should notice the mechanism that when input delay exists, perturbations imposed on the system are sequentially compensated with each delay interval due to the causality principle and an ultimate bound will be caused inevitably. In Fig. 8, the uncertain model dynamics varies to $\dot{x}_3 = \frac{0.4+\sin(x_2x_3)}{\cos(x_3)+2} + (0.2\sin(5t)+1)u(t-\mathcal{D}(t))$ compared with (78), and the positive term $\frac{0.4}{\cos(x_3)+2}$ obviously renders the system instable and forward complete property in *Definition* 2.1 isn't valid any more. The results in Fig. 8 show that the strong anti-disturbance ability of CNTSM prevents the state escape and a fast convergence is achieved simultaneously compared with the state feedback control.

VI. Conclusion

In this paper, a class of invertible nonlinear system under the co-existence of time-varying input delay and perturbations is considered. We develop a full order Lipschitz continuous and chattering-free terminal SMC along with the infinite-dimensional backstepping predictor to stabilize this system. The input-to-state stability for target system is proved rigorously including internal dynamics while the steady state error and anti-disturbance ability under the proposed method are both superior to the conventional feedback control.

Appendix I

First, select the Lyapunov candidate of external state as $V_1(\zeta) = \zeta^T P\zeta$ where P is a positive symmetric matrix and use the Rayleigh-Ritz inequality, the following relationships hold

$$\alpha_1(\|\zeta\|_2) \le V_1(\zeta) \le \alpha_2(\|\zeta\|_2) \tag{85}$$

$$\alpha_1(\|\zeta\|_2) \triangleq \lambda_{\min}(P)\|\zeta\|_2^2, \quad \alpha_2(\|\zeta\|_2) \triangleq \lambda_{\max}(P)\|\zeta\|_2^2 \tag{86}$$

where α_1, α_2 are class $\mathcal{K}_\infty$ functions and $\lambda_{\min}(P), \lambda_{\max}(P)$ are the minimum and maximum eigenvalues of P respectively. Take the derivative of $V_1(\zeta)$ along the external dynamics in (9) with the virtual control $\nu = K\zeta$ which renders $\lambda_{\min}(Q) > 2k_\vartheta$ for the Lyapunov equation $(A_c + B_cK)^T P + P(A_c + B_cK) = -Q$.

$$\begin{aligned}\dot V_1 &= \zeta^T[(A_c + B_cK)^T P + P(A_c + B_cK)]\zeta + 2\zeta^T PB_c\varrho \\ &\le -\lambda_{\min}(Q)\|\zeta\|_2^2 + 2k_\vartheta\|PB_c\|_2\|\zeta\|_2^2 \\ &\quad + 2k_\vartheta\|PB_c\|_2\|\zeta\|_2\|\eta\|_2 + 2\|\zeta\|_2\|PB_c\|_2(\delta_\vartheta + \bar\varrho_\vartheta) \\ &\le -(\lambda_{\min}(Q) - 2k_\vartheta)\|\zeta\|_2^2 + 2\|\zeta\|_2\|PB_c\|_2(\bar\delta_\vartheta + \bar\varrho_\vartheta) \\ &\le -\iota_1(\lambda_{\min}(Q) - 2k_\vartheta)\|\zeta\|_2^2, \ \iota_1 \in (0,1) \\ \forall|\zeta\|_2 &\ge \varsigma_1(\bar\delta_\vartheta + \bar\varrho_\vartheta), \quad \varsigma_1 = \frac{2\,\|PB_c\|_2}{(\lambda_{\min}(Q) - 2k_\vartheta)(1-\iota_1)}\end{aligned} \tag{87}$$

where $\bar\delta_\vartheta = \delta_\vartheta + k_\vartheta\eta_M$ (see *Assumption* 3.2). Consequently, there exists a class $\mathcal{KL}$ function β_1 and finite $t_1 \in \mathbb{R}^+$ independent of t_0 such that external state ζ is ultimately bounded by

$$\|\zeta(t)\| \le \beta_1(\|\zeta(t_0)\|, t - t_0), \ t_0 \le \forall t \le t_0 + t_1 \tag{88}$$

$$\|\zeta(t)\| \le \alpha_1^{-1}(\alpha_2(\varsigma_1(\bar\delta_\vartheta + \bar\varrho_\vartheta))) \triangleq \bar\Upsilon_{pert}, \ \forall t > t_0 + t_1 \tag{89}$$

Next, according to the converse Lyapunov theorem (see [16]), we obtain (90,91) where $V_{\eta 2}$ is the Lyapunov candidate of internal dynamics and $\sigma_i(i = 1, \cdots, 4)$ are class $\mathcal{K}$ functions on $\mathcal{D}_{\eta 1}$.

$$\sigma_1(\|\eta\|_2) \le V_{\eta 2}(\eta) \le \sigma_2(\|\eta\|_2), \ \eta \in \mathbb{R}^{n-r} \tag{90}$$

$$\frac{\partial V_{\eta 2}}{\partial \eta}\varpi(0,\eta,0) \le -\sigma_3(\|\eta\|_2), \quad \|\frac{\partial V_{\eta 2}}{\partial \eta}\|_2 \le \sigma_4(\|\eta\|_2) \tag{91}$$

Moreover, $\sigma_j(\|\eta\|_2) \triangleq p_j\|\eta\|_2^2\,(j = 1, 2, 3)$, $\sigma_4(\|\eta\|_2) = p_4\|\eta\|_2$ are satisfied attributed to the exponential stable hypothesis. According to *Assumption* 3.2 there exists a Lipschitz constant L for ϖ w.r.t. ζ, d and analogously take the derivative along internal dynamics yields

$$\begin{aligned}\dot V_2 =& \frac{\partial V_{\eta 2}}{\partial \eta}\varpi(0,\eta,0) + \frac{\partial V_{\eta 2}}{\partial \eta}[\varpi(\zeta,\eta,d) - \varpi(0,\eta,0)] \\ \le& - p_3\|\eta\|_2^2 + p_4L\|\eta\|_2(\|\zeta\|_2 + \|d\|_2) \\ \le& - \iota_2 p_3\|\eta\|_2^2, \ \iota_2 \in (0,1), \ \frac{p_4L(\|\zeta\|_2 + \|d\|_2)}{p_3(1-\iota_2)} \le \|\eta\|_2 \le \eta_M\end{aligned} \tag{92}$$

Utilizing the boundedness theorem (see [16]), select $r > 0$ such that $\mathcal{D}_{\eta 2} = \{\eta : \|\eta\| < r\} \subset \mathcal{D}_{\eta 1}$, if the following inequalities are satisfied (which means there are limitations on perturbation and norm of the initial state $\eta(t_0)$), i.e., $\|\eta(t_0)\|_2 \le \sigma_2^{-1}(\sigma_1(r))$ and

$$\frac{p_4L(\|\zeta\|_2 + \|d\|_2)}{p_3(1-\iota_2)} \le \frac{p_4L(\bar\Upsilon_{pert} + \bar d)}{p_3(1-\iota_2)} \triangleq \bar\chi < \sigma_2^{-1}(\sigma_1(r)) \tag{93}$$

then the internal state η can be bounded by a class of $\mathcal{K}$ functions for some finite $t_2 > 0$ which is dependent on $\eta(t_0)$ and $\bar\chi$

$$\begin{aligned}&\|\eta(t)\|_2 \le \beta_2(\|\eta(t_0)\|_2, t - t_0) + \sigma_1^{-1}(\sigma_2(\bar\chi)) \\ &\|\eta(t)\|_2 \le \iota_3\bar\chi + \sigma_1^{-1}(\sigma_2(\bar\chi)), \ \forall t \ge t_0 + t_1 + t_2\end{aligned} \tag{94}$$

where the term $\iota_3\bar\chi$ can be arbitray small postive value due to the property of functions $\mathcal{KL}$. Finally, whole state $z = \mathrm{col}(\eta, \zeta)$ is bounded by the following inequality for $\forall t \ge t_0 + t_1 + t_2$ and the maximum perturbation that the system can tolerate is also given.

$$\bar\delta_\vartheta + \bar\varrho_\vartheta + \bar d < \frac{p_3(1-\iota_2)}{\varsigma_1 p_4 L}\alpha_2^{-1}(\alpha_1(\sigma_2^{-1}(\sigma_1(r)))) \tag{95}$$

$$\|z(t)\|_2 \le \|\eta(t)\|_2 + \|\zeta(t)\|_2 = \bar\Upsilon_{pert} + \iota_3\bar\chi + \sigma_1^{-1}(\sigma_2(\bar\chi)) \tag{96}$$

Thus, *Proposition* 1 is proved completely.

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